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A criterion for the growth of cracks with bonds in the end zone☆

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Abstract

A fracture criterion which takes account of the work done in the deformation of bonds in the end zone of a crack is proposed for analysing the quasistatic growth of a crack with bonds in the end zone. The energy condition that the deformation energy release rate at the crack tip is equal to the rate of deformation energy consumption by the bonds in the end zone of the crack (the first fracture condition) corresponds to the state of limit equilibrium of the crack tip. The rupture of bonds at the trailing edge of the end zone is determined by the condition for their limiting traction (the second fracture condition). Starting from these two conditions, the processes of subcritical and quasistatic crack growth are considered for the case of a rectilinear crack at interface of materials and the two basic fracture parameters, the critical external load and the size of the end zone of the crack in the state of limit equilibrium, are determined. Analytical expressions are obtained for the deformation energy release rate at the crack tip and the rate of deformation energy consumption by the bonds and, also, the dependences of the critical external load and size of the end zone of the crack on the crack length in the case of a rectilinear crack in a homogeneous body with bond tractions which are constant and independent of the external load. The limit cases of a crack which is filled with bonds and a crack with a short end zone are considered.

1. Introduction

The Griffith fracture criterion is used for materials in which the fracture process is localized in a small domain close to the crack tip (the fracture process zone) and the interaction of the newly formed surfaces of the crack can be neglected. In the case of a small fracture process zone, the state of limit equilibrium of a crack is completely defined by the critical stress intensity factor or by the modulus of cohesion of the material.[1,2] In structurally inhomogeneous materials (adhesive compounds, composites and geomaterials), when there are domains with a disrupted structure close to a crack, and physical fields and aggressive media act on the fracture process, quite a large part of the crack becomes involved in the fracture process and different fracture mechanisms can occur when the size of the end zone of a crack changes. In such cases, the fracture zone can be considered as a certain layer of finite length (the end zone) which is adjacent to the crack and contains material with partially ruptured bonds between its individual structural elements. One of the ways of modelling such a layer involves treating it as a part (continuation) of the crack and the explicit application of cohesive forces to the crack surfaces in the end zone which restrain the opening of the crack. The tip (front) of such a modified crack coincides with the leading edge of the end zone. As a rule, the magnitude of

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the cohesive forces σ depends on the opening of the crack u in the end zone

$$\sigma = \sigma(u) \tag{1.1}$$

and the environmental conditions (the presence of physical fields, aggressive media, etc.).

The physical nature of the cohesive forces depends on the structure of the material and the dimensions of the crack and the end zone. Direct intermolecular interaction predominates at short distances from the crack tip and “mechanical bonds” make a significant contribution at relatively large distances from the crack tip. For example, in composite and nanocomposite materials, such bonds may consist of reinforcing fibres or bundles of nanotubes respectively and, in polymer adhesive compounds, of the monomer units of polymer chains which connect the surfaces of the crack.

Two approaches are mainly used for the mathematical description of the cohesive forces in the end zone of a crack: 1) the application of discrete values of the cohesive forces to the surfaces of the crack which simulate the presence of fibres or particles which restrain the opening of the crack,[3,4] and 2) consideration of a continuous distribution of cohesive forces in the end zone of the crack.[5–11] Models of the first type are used, for example, in cases when the distance between the fibres is comparable with the characteristic size of the crack or the end zone.

In the treatment of the model of a crack with cohesive forces in the end zone, it is necessary to use a criterion for the crack growth which enables one to determine the conditions for the advance of the trailing edge of the end zone as well as of the crack tip.

A force condition of the form [9]

$$K_\infty - K_b = K_{Ic} \tag{1.2}$$

where K_∞ is the stress intensity factor due to the action of the external loads, K_b is the stess intensity factor due to the presence of the reinforcing fibres and K_{Ic} is the fracture toughness of the material, which corresponds to the mechanical deformation of the bonds in a small zone close to the crack (for example, the fracture toughness of the matrix material), is conventionally used as the first condition of such a criterion for the crack growth.

A choice of the condition for the critical opening of a crack at the trailing edge of the end zone (or the critical strain condition).

$$u(\sigma_*, L, d) = \delta_{cr} \tag{1.3}$$

where σ_* is the critical external load, and L and d are the characteristic linear dimensions of the crack and the end zone respectively, is possible as the second condition for the growth of a crack with an end zone.

Another possibility, for example, consists of using the conditions for the critical stresses to be reached at the crack tip σ_{th}

$$\sigma(\sigma_*, u, L, d) = \sigma_{th} \tag{1.4}$$

instead of (1.2). In this case, the stress at the crack tip means the mean value of the stress on a certain characteristic segment close to the tip.[12,13]

Note that, when conditions (1.2) or (1.4) are used, the work done in deforming the bonds in the end zone turns out to be outside the framework of the fracture criterion.

In this paper, a fracture criterion with an energy condition for the advance of the crack tip, which takes account of the work done in deforming the bonds in the end zone of the crack, is used to analyse the limit equilibrium of a crack with an end zone. This condition is based on the equality of the deformation energy release rate at the crack tip and the energy consumption rate in deforming the bonds in the end zone of the crack. The kinematic Eq. (1.3) is used to determine the advance of the trailing edge of the end zone of the crack.

2. Fracture criterion for a crack with bonds in the end zone

Consider an elastic body with a plane crack of area S and an end zone of area S_d, $S_d \subset S$. External loads $T_i^e(s)$ are applied to the boundary of the body, and the displacements of the points s of the body surface $u_i^e(s)$ are specified, where $i = 1, 2$ or $i = 1, 2, 3$ in the two-dimensional and three-dimensional cases respectively. We will assume that the surfaces of the crack interact in the end zone S_d adjacent to the crack front so as to restrain the opening of the crack. In order to describe the interaction of the surfaces of the crack, we will assume that there are bonds in the end zone S_d between the surfaces of the crack and that the law of deformation of these bonds, which is non-linear in the general case, is specified. Tractions $Q_i(s)$ arise under the action of external loads in the bonds joining the surfaces of the crack, and corresponding tractions of opposite sign are applied to the surfaces of the crack.

Assuming that the opening of the crack $u_i(s)$ and the stresses (tractions) in the bonds $Q_i(s)$ are known and that the body surface in the end zone of the crack S_d is loaded by the tractions

$$T_i^d(s) = -Q_i(s) \tag{2.1}$$

we write the following expression for the total energy of the body with loads $T_i^e(s)$ on the outer surface of the body S_e and with the loads (2.1) on the end zone of the crack S_d

$$F = \Pi + U + \Sigma_m \tag{2.2}$$

Here U is the potential energy of the deformation of the bonds in the end zone of the crack and Π is the total potential energy of the elastic body

$$\Pi = W - A \tag{2.3}$$

where W is the potential energy of deformation and A is the work of the external forces.

The last term in equality (2.2) is the surface energy of the matrix material

$$\Sigma_m = G_m S; \quad G_m = 2c_m\gamma_m, \quad c_m = 1 - c_f \tag{2.4}$$

where c_f is the specific concentration of fibres in the composite and γ_m is the specific surface energy of the matrix material.

When there are no mass forces

$$\Pi = \int_V w(\varepsilon_{ij})dV - \int_{S_e} T_i^e(s)u_i^e(s)dS - \int_{S_d} \{T_i^d(s)u_i^+(s) - T_i^d(s)u_i^-(s)\}dS \tag{2.5}$$

where $w(\varepsilon_{ij})$ is the density of the deformation energy of the elastic body in the volume V, ε_{ij} are the components of the strain tensor, $u_i^e(s)$ are the displacements of the points of the outer surface of the body S_e, and $u_i^+(s)$ and $u_i^-(s)$ are the displacements of the surfaces of the crack in the end zone S_d.

We define the components of the opening of the crack in the end zone as

$$u_i^d(s) = u_i^+(s) - u_i^-(s) \tag{2.6}$$

and, taking account of relations (2.1) and (2.6), we can write expression (2.5) in the form

$$\Pi = \int_V w(\varepsilon_{ij})dV - \int_{S_e} T_i^e(s)u_i^e(s)dS + \int_{S_d} Q_i^d(s)u_i^d(s)dS \tag{2.7}$$

In equality (2.2), we represent the potential energy of the deformation of the bonds in the end zone of the crack in the form

$$U = \int_{S_d} \Phi(u^d)dS \tag{2.8}$$

where $\Phi(u^d)$ is the strain potential energy density of the bonds in the end zone of the crack. We obtain an energy criterion for the equilibrium of the crack in the case of a small increment in its size (area, length) from expressions (2.2) and (2.3)

$$\delta F = \delta(\Pi + U + \Sigma_m) = 0 \tag{2.9}$$

that is, the change in the potential energy of the system is determined by the energy required to form the new crack surface and the deformation of the bonds in the end zone of the crack.

Note that the definition of the increment in the potential energy when a dimension of the crack changes in accordance with equality (2.9) assumes, in the general case, the existence of infinite stresses at the crack tip (along the front). The energy condition (2.9) is identical in form to the Griffith criterion and, when treating two-parameter fracture models, is the only necessary fracture condition.[12,14,15] When treating models of a crack with an end zone, this is expressed in the fact that it is impossible to determine the critical fracturing load starting solely from this criterion.[10,11] It can be shown in the limiting case of a crack with a short end zone that condition (2.9) is equivalent to the conditions for the stresses at the crack tip to be finite.[10]

To analyse the limit equilibrium of a crack with an end zone, it is necessary to consider an additional critical condition and it is possible to use a treatment of the conditions of critical bond traction, of a critical load on a bond, and of the critical opening of the crack as this additional condition. Conditions of this type are, in essence, force or kinematic sufficient conditions for fracture.

We shall next consider a condition for the critical opening of a crack at the trailing edge of the end zone as an additional sufficient condition for fracture. We will assume that the bonds rupture at the trailing edge of the end zone (or at a certain point on the contour of the end zone in the three-dimensional case) occurs when the condition

$$u(s_0) = \delta_{cr} \tag{2.10}$$

is satisfied, where $u(s_0)$ is the opening of the crack at the trailing edge of the end zone and δ_{cr} is the limit length of a bond.

Eqs. (2.9) and (2.1) are a fracture criterion for a crack with bonds in the end zone. The combined solution of these equation enables one, when the dimensions of the crack and the characteristics of the bonds are specified, to determine two basic parameters, the critical external load and the size of the end zone in the state of limit equilibrium of the crack.

3. An interface Crack the Fracture criterion

We will now consider the application of the fracture criterion (2.9), (2.10) in the case of the uniaxial tension of a two-dimensional domain containing a rectilinear crack at the interface of half-planes with different mechanical properties, the axis of which is perpendicular to the direction of application of the load.

In the two-dimensional problem, expression (2.9) for a rectilinear crack, occupying a segment $|x| \le l$, $y = 0$, with end zones of size d (Fig. 1), is written, by analogy with the case considered earlier[10,11], as:

$$-\frac{\partial \Pi}{\partial l} = \frac{\partial U}{b \partial l} + G_m \tag{3.1}$$

We will introduce the notation

$$G_{\text{tip}}(d, l) = -\frac{\partial \Pi}{\partial l}, \quad G_{\text{bond}}(d, l) = \frac{\partial U}{b \partial l} + G_m \tag{3.2}$$

where $G_{\text{tip}}(d, l)$ is the deformation energy release rate (the energy flux at the crack tip[1]), $G_{\text{bond}}(d, l)$ is the energy consumption rate in the deformation of the bonds in the end zone of the crack and b is the thickness of the body.

The expression for the deformation energy release rate in the case of a crack at the interface of different materials still holds when there are bonds in the end zone of the crack since the effect of the bonds is expressed in the application of the loads $T_i^d(s)$ to the crack surfaces in end zone. Hence, regardless of the form of the law of deformation of the

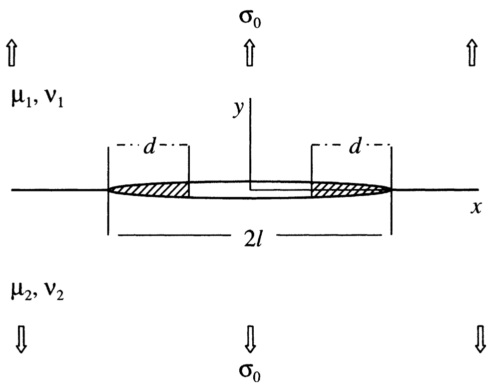

Fig. 1.

bonds, the deformation energy release rate is given by the expression[16]

$$G_{\mathrm{tip}}(d,l) = \left(\frac{k_1+1}{\mu_1}+\frac{k_2+1}{\mu_2}\right)\frac{K_I^2+K_{II}^2}{16\,\mathrm{ch}(\pi\beta)};\quad \beta = \frac{\ln\alpha}{2\pi},\quad \alpha = \frac{\mu_2 k_1+\mu_1}{\mu_1 k_2+\mu_2} \tag{3.3}$$

where $k_{1,2} = 3 - 4\nu_{1,2}$ in the case of plane strain or $k_{1,2} = (3-\nu_{1,2})/(1+\nu_{1,2})$ in the case of a plane stress state, $\nu_{1,2}$ and $\mu_{1,2}$ are Poisson's ratios and the shear moduli of the materials of the subdomains 1 ($y>0$) and 2 ($y<0$) (see Fig. 1). The stress intensity factors (SIFs) K_{I} and K_{II} are determined taking account of the SIFs from the loads on the outer surface of the body $K_{i,\infty}$ and the SIFs from the loads in the end zone of the crack $K_{i,b}$[10,11]

$$K_i = K_{i,\infty} - K_{i,b},\quad i = \mathrm{I, II} \tag{3.4}$$

Using relation (2.8), we define the strain potential energy of the bonds in the end zone of the crack as follows:

$$U = b\int_{l-d}^{l}\Phi(u)dx,\quad \Phi(u) = \int_0^{u(x)}\sigma(u)du,\quad u(x) = \sqrt{u_x^2+u_y^2},\quad \sigma(u) = \sqrt{q_x^2+q_y^2} \tag{3.5}$$

where $\Phi(u)$ is the density of the potential energy of the deformation of the bonds in the end zone of the crack, $u_{x,y}(x)$ are the components of the opening of the crack in the end zone, $\sigma(u)$ is the modulus of the traction vector in the bonds, which is determined by the law of deformation of the bonds, and $q_{y,x}(u)$ are the normal and tangential components of the bond traction.

Note that ($i=x$ for the tangential component and $i=y$ for the normal component)

$$q_i = \frac{\partial\Phi(u)}{\partial u_i} = \sigma(u)\frac{\partial u}{\partial u_i} = \kappa(u)u_i,\quad \kappa(u) = \frac{\sigma(u)}{u},\quad c(u) = \frac{1}{\kappa(u)} \tag{3.6}$$

where $\kappa(u)$ and $c(u)$, the effective stiffness and compliance of the bonds respectively, are the same for the tangential and normal strains.

Using relation (3.6), the strain of the bonds can be written as

$$q_y(x) - iq_x(x) = c(u)(u_y(x) - iu_x(x)),\quad i^2 = -1 \tag{3.7}$$

The bond tractions $q_{y,x}(u)$ are determined in the end zone of the crack when $l-d \le |x| \le l$. In the case of a composite material with reinforcing fibres, the relative concentration of which is c_f, the tractions in the end zone are related to the tractions of the reinforcing fibres t_i as follows:

$$q_i = t_i c_f \tag{3.8}$$

and, when treating the adhesive layer between the materials, we assume that $q_i = t_i$.

Substituting expression (3.5) into the second equality of (3.2), we obtain

$$G_{\mathrm{bond}}(d,l) = \frac{\partial U}{b\partial l} + G_m = \frac{\partial}{\partial l}\int_{l-d}^{l}\left(\int_0^{u(x)}\sigma(u)du\right)dx + G_m \tag{3.9}$$

Differentiation with respect to the upper and lower limits of integration in expression (3.9) for the rate of energy consumption in deforming the bonds corresponds to the assumption that there is a change in the size of the end zone as a result of the rupture of bonds at the trailing edge of the end zone (when $x_0 = l-d$) and a simultaneous advance of the crack tip. In such a case, the condition for the end zone of the crack to be autonomous can be satisfied.

Differentiating with respect to the upper and lower limits in Eq. (3.9), we obtain

$$G_{\mathrm{bond}}(d,l) = \int_{l-d}^{l}\left(\frac{\partial u(x)}{\partial l}\sigma(u)\right)dx + G_m + G_c - G_b;\ G_c = \int_0^{u(l)}\sigma(u)\sigma u,\ G_b = \int_0^{u(l-d)}\sigma(u)du \tag{3.10}$$

and, in the state of limit equilibrium of the crack $u(l-d) = \delta_{\mathrm{cr}}$ (according to condition (2.10)). On choosing a crack model in such a way that the opening at the crack tip is equal to zero $u(l)=0$, we have $G_c = 0$.

Taking account of relations (3.6) and (3.7), expression (3.10) can be written as

$$G_{\text{bond}}(d, l) = \int_{l-d}^{l} \left(\frac{\partial u_y(x)}{\partial l} q_y(u) + \frac{\partial u_x(x)}{\partial l} q_x(u) \right) dx - G_b + G_m \tag{3.11}$$

The quantity G_b is the deformation energy density released during the rupture of bonds at the trailing edge of the end zone $x_0 = l - d$.

In the case of a homogeneous material or an adhesive layer joining the different materials, we assume that

$$G_m = G_b = \int_0^{\delta_{cr}} \sigma(u) du \tag{3.12}$$

and expression (3.1) is identical to that obtained earlier (Ref. 11, formula (2.9)).

In treating a composite material with a matrix with a low fracture toughness (fracture toughness is soley determined by the bonds in the end zone), we assume that $G_b \gg G_m$. For this special case, it will be shown below that

$$G_{\text{bond}}(d, l) \to 0, \quad G_{\text{tip}}(d, l) \to 0 \text{ при } d/l \to 0$$

and expression (3.1) becomes the condition for the stresses at the crack tip to be finite.

Satisfaction of the necessary and sufficient conditions

$$G_{\text{tip}}(d, l) = G_{\text{bond}}(d, l), \quad u(x_0) = [u_x^2(x_0) + u_y^2(x_0)]^{1/2} = \delta_{cr} \tag{3.13}$$

corresponds to the state of limit equilibrium of the crack tip and the trailing edge of the end zone of the crack. The parameter δ_{cr} is defined by the properties of the bonds in the end zone of the crack and can also depend on the scale of the crack (for example, when the type of bonds changes as the crack grows).

From the simultaneous solution of Eq. (3.13) it is possible to determine the size of the end zone d_{cr} and the critical external stress σ_{cr} in the state of limit equilibrium of the crack. The deformation energy consumption rate $G_{\text{bond}}(d_{cr}, l)$, obtained from the simultaneous solution of Eq. (3.13) is an energy characteristic of the adhesive fracture toughness, $G_{cr} = G_{\text{bond}}(d_{cr}, l)$, and the quantity G_{cr} does not remain constant when the crack length changes. After the critical external load has been determined, the critical stress intensity factor and the energy flux at the crack tip, due to the external load σ_{cr}, can also be determined.

The following conditions are now considered.

Condition 1. a) $G_{\text{tip}}(d, l) \geq G_{\text{bond}}(d, l)$, b) $G_{\text{tip}}(d, l) < G_{\text{bond}}(d, l)$

Condition 2. a) $u(l - d) < \delta_{cr}$, b) $u(l - d) \geq \delta_{cr}$

During monotonic loading of a body, when the initial dimensions of the crack and its end zone are specified, it is possible to distinguish the equilibrium and quasistatic growth of the crack.

When Conditions 1a and 2a are satisfied, the crack tip advances with a simultaneous increase in the length of the end zone without bonds rupture. This stage in the crack growth can be considered as the adaptability of the crack to a specified level of external loads (subcritical crack growth).

The advance of the crack tip with the simultaneous rupture of bonds at the trailing edge of the end zone (quasistatic crack growth) occurs when Conditions 1a and 2b are simultaneously satisfied. When Conditions 1b and 2b are satisfied, bonds rupture occurs without an advance of the crack tip and the size of the end zone decreases, tending to a limit value for a given load level.

Within the framework of the model being considered, the position of the end zone and the crack tip do not change when Conditions 1b and 2a are simultaneously satisfied.

Hence, the magnitude of the external load σ_0 and the bond parameters determine the nature of the fracture: an advance of the crack tip with a growth of the end zone; a decrease in the size of the end zone without an advance of the crack tip, and an advance of the crack tip with the simultaneous rupture of bonds at the trailing edge of the end zone.

The stress and strain distributions in the end zone of the crack are required in order to analyse the limit equilibrium state of cracks using relations (3.13). When account is taken of the functional relation between these quantities (see, for example, equality (3.7)), the problem of determinig the strains and stresses in the end zone of a crack in a homogeneous material or at a boundary where half-planes join reduces to the numerical solution of an integral or integrodifferential equation.[7–11]

If the bond tractions are independent of the opening of the crack and are constants, the known analytical expressions for the opening of the crack in the end zone can be used to analyse the limit equilibrium of the crack. Analytical expressions for the energy characteristics of the crack are obtained below for a special case of a crack in a homogeneous plane with constant bond tractions, and a procedure for using fracture criterion (3.13) to analyse the crack growth is considered.

4. Fracture criterion, a homogeneous body

We will now consider the application of fracture criterion (3.13) using the example of the problem of the uniaxial tension of a homogeneous plane with a rectilinear crack of length $2l$ (see, Fig. 1, $\mu_1 = \mu_2 = \mu_0$, $\nu_1 = \nu_2 = \nu_0$). A tensile stress $\sigma_y = \sigma_0$ is applied at the remote external boundary, and the bond tractions $q_y(x) = P_0$ in the end zone of the crack are independent of the opening of the crack and constant along the end zone. There are no tangential loads on the crack surfaces. The displacements of the upper surface of the crack are given by the expressions[6]

$$\begin{aligned} u(x) &= \frac{1}{E}\left(2\sigma_0 - \frac{4P_0}{\pi}\arccos\frac{h}{l}\right)\sqrt{l^2 - x^2} + \frac{P_0}{\pi E}[(x-h)F(l,x,h) - (x+h)F(l,x,-h)] \\ u_y(x) &= 2u(x), \quad u_x(x) = 0 \end{aligned} \tag{4.1}$$

where E is the modulus of elasticity ($E = E_0 = 2\mu_0(1+\nu_0)$ for a plane stress state or $E = E_0/(1-\nu_0^2)$ for a state of plane strain), ν_0 is Poisson's ratio, $h = l - d$ (Fig. 1) and the influence function $F(l, x, \xi)$ is given in Ref. 6 by expression (1.8).

The release deformation energy rate (3.3) is defined, in the case of a homogeneous body, as follows (plane stress state)[1]:

$$G_{\text{tip}}(d,l) = \frac{K^2}{E}; \quad K = K_\infty - K_b = \sigma_0\sqrt{\pi l}\left(1 - \frac{2}{\pi\sigma_0}\int_{1-d/l}^{1}\frac{q_y(t)}{\sqrt{1-t^2}}dt\right) \tag{4.2}$$

For constant bond tractions P_0, we obtain

$$G_{\text{tip}}(d,l) = G_f(1 - Z_0\arccos(1-t))^2; \quad G_f = \frac{\sigma_0^2\pi l}{E}, \quad Z_0 = \frac{2P_0}{\pi\sigma_0}, \quad t = \frac{d}{l} \tag{4.3}$$

The expression for the energy rate consumption in deforming the bonds (3.9) has the form

$$G_{\text{bond}}(d,l) = 2P_0\left[\frac{\partial}{\partial l}\int_{l-d}^{l} u(x)dx\right] + G_m \tag{4.4}$$

where G_m is the effective surface energy of the matrix and, in the case of an adhesive layer, $G_m = G_b = P_0\delta_{\text{cr}}$ (see relation (3.12)).

Evaluating the integral and the derivative on the right-hand side of equality (4.4), we obtain

$$\begin{aligned} G_{\text{bond}}(d,l) &= 2G_f\varphi(t) + G_m \\ \varphi(t) &= Z_0\{A(t) - B(t) - Z_0[A(t)[A(t) - 2B(t)] - 2C(t)]\}, \quad t = d/l \end{aligned} \tag{4.5}$$

where

$$A(t) = \arccos(1-t), \quad B(t) = \sqrt{2t - t^2}, \quad C(t) = (1-t)\ln(1-t) \tag{4.6}$$

Equating the expressions for $G_{tip}(d, l)$ (4.3) and $G_{bond}(d, l)$ (4.5), we obtain the equation corresponding to the first conditions of (3.13)

$$2Z_0\left\{[2A(t)-B(t)]-Z_0\left[\frac{3}{2}A^2(t)-2A(t)B(t)-2C(t)\right]\right\}+2\eta R_0Z_0^2-1 = 0 \tag{4.7}$$

We obtain the equation corresponding to the second condition of (3.3) starting from expression (4.1), when $x = l - d$

$$\frac{B(t)}{Z_0}-[A(t)B(t)+C(t)] = R_0 \tag{4.8}$$

In Eqs. (4.7) and (4.8)

$$R_0 = \frac{\pi E\delta_{cr}}{8P_0l}, \quad \eta = \frac{G_m}{G_b}, \quad G_b = P_0\delta_{cr} \tag{4.9}$$

The system of non-linear algebraic Eqs. (4.7), (4.8) corresponds to the conditions of the limit equilibrium of a crack with an end zone (3.13) and contains two unknown quantities: $Z_0 = 2P_0/(\pi\sigma_{cr})$ and $t = d_{cr}/l$. The solution of this system is completely determined by the parameters R_0 and η, and it enables us to obtain the relations between the quantities σ_{cr} and d_{cr} and the length of the crack during its quasistatic growth. When there is no solution of system of Eqs. (4.7), (4.8), one of the fracture processes described at the end of Section 3 can occur.

A clear representation of the solution of system of Eqs. (4.7), (4.8) can be obtained from an analysis of Fig. 2, where graphs of the normalized deformation energy release rate $\tilde{G}_{tip} = G_{tip}/G_f$ and the normalized energy consumption rate in deforming the bonds $\tilde{G}_{bond} = G_{bond}/G_f$ against the relative length of the end zone of the crack $t = d/l$ are shown by the solid lines for a value of the relative length of the crack $\lambda = (l/d_0) = 0.648$ and $\eta = 1$; the parameter d_0 is defined by the expression

$$d_0 = \frac{\pi E\delta_{cr}}{8P_0} \tag{4.10}$$

The values of G_{tip} and G_{bond} are given by formulae (4.3) and (4.5) respectively and normalized with the quantity G_f determined for $\sigma_0 = \sigma_{cr} = 1.682P_0$, – the critical load for the crack being considered. The point of intersection of the graphs of $\tilde{G}_{tip}$ and $\tilde{G}_{bond}$ determines the relative magnitude of the end zone of the crack $t_{cr} = d_{cr}/l = 0.284$ (normalization of the quantity d_0 gives $d_{cr}/d_0 = 0.184$). The relation $u_0 = 2u(1 - t)/\delta_{cr}$ is shown by the dashed curve and the function $u(x)$ is given by formula (4.1). At points 1 and 2 $u_0 = 1$, which corresponds to the satisfaction of the second condition of (3.13) but only point 1 corresponds to the limit equilibrium of the crack and, in this case, both conditions of (3.13) are satisfied.

We will now consider the growth of a crack with an end zone, assuming that the initial cut, a crack of length $2l_0$, has no bonds. When the load σ_0 is increased monotonically, two end zones, each with a dimension d, are formed near

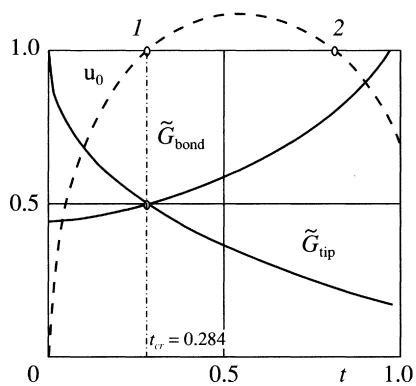

Fig. 2.

the crack tips. Then,

$$t = d/(l_0 + d) \tag{4.11}$$

Note that Condition 1a is satisfied for each value of the length of the end zone d and the external load σ_0 and the opening of the crack at the trailing edge of the end zone does not exceed the critical value (Condition 2a, subcritical crack growth). We obtain the dependence of the magnitude of the external load $\sigma_0 = \sigma_G$ on the size of the end zone from condition (4.7) by treating it as an equation in the parameter Z_0:

$$Z_0^2\left[\frac{3}{2}A^2(t) - 2A(t)B(t) - 2C(t) - \eta R_0\right] - Z_0[2A(t) - B(t)] + \frac{1}{2} = 0 \tag{4.12}$$

Using the definition of the parameter Z_0 (4.3), from Eq. (4.12) we obtain

$$\frac{\sigma_G}{P_0} = \frac{2}{\pi Z_0(t, \eta, R_0)} = \frac{2}{\pi}[2A(t) - B(t) + \sqrt{A^2(t) + B^2(t) + 4C(t) + 2\eta R_0}] \tag{4.13}$$

The parameter t is defined by formula (4.11).

According to Eq. (4.13), subcritical crack growth occurs when the external load is increased, the size of the end zone of the crack becomes larger and the critical value is reached when the second condition of (3.13) is satisfied. In order to maintain the process of quasistatic crack growth for longer, it is necessary to reduce the external load such that Eqs. (4.7) and (4.8) are satisfied.

Curves for the subcritical growth of a crack from the initial cut, which are determined by expression (4.13), and their envelope, corresponding to the state of limit equilibrium, are shown in Fig. 3 for values of the parameter $\eta = 0$ and $\eta = 1$; the parameter $\lambda = (l_0 + d)/d_0$ is the relative length of the crack and $\lambda_j = l_0/d_0$ ($j = 1, \ldots, 5$) is the relative size of the initial cut, which has no bonds.

Each of the curves for the subcritical crack growth starts from the point $\lambda = \lambda$, $\sigma_G = 0$ and terminates on the limit equilibrium curve when $\lambda = (l_{0j} + d_{\mathrm{cr}})/d_0$, $\sigma_G = \sigma_{\mathrm{cr}}$. Note that, in the case of α matrix with low fracture toughness ($\eta = 0$), the crack growth accompanying subcritical crack development (the size of the end zone) is greater than when

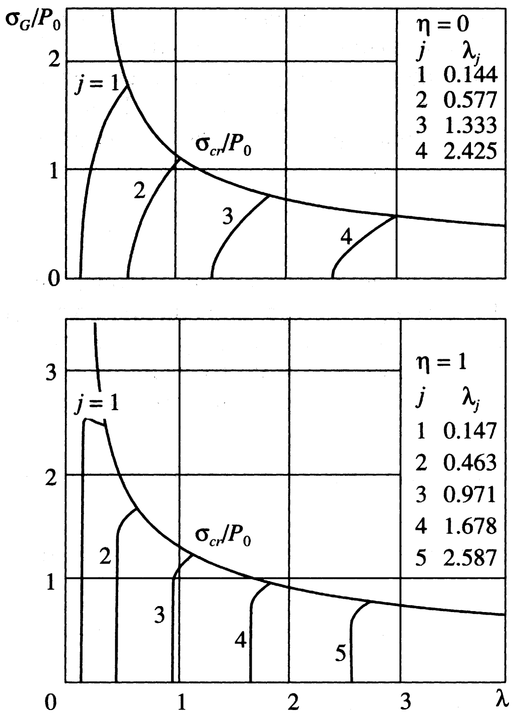

Fig. 3.

$\eta = 1$. A non-monotonic change in the external load during the growth of a crack from a short cut (see the case when $\eta = 1$, $\lambda_1 = 0.147$) is characteristic when account is taken of the fracture toughness of the matrix $\eta \geq 1$.

We will now consider a crack which is filled with bonds. In this case, when the external load is increased monotonically, it is possible that either Conditions 1a and 2a are satisfied (quasistatic crack growth without the rupture of the bonds) or Conditions 1b and 2b are satisfied (rupture of the bonds without any advance of the crack tip). We shall assume that the size of the crack is such that Conditions 1b and 2b are satisfied. The dependence of the magnitude of the external load $\sigma_0 = \sigma_U$ on the size of the end zone is determined in this case by Condition 2b. Using the definition of the parameter Z_0 (4.3), from Eq. (4.8) we obtain

$$\frac{\sigma_U}{P_0} = \frac{2}{\pi}\left[\frac{R_0 + A(t)B(t) + C(t)}{B(t)}\right], \quad 1 \leq t < \frac{d_{cr}}{l} \tag{4.14}$$

Expression (4.14) corresponds to the assumption that, in the case of a crack which is filled with bonds, rupture of the bonds starts from the centre of the crack. According to expression (4.14), when the magnitude of the external load is changed, rupture of the bonds at the trailing edge of the end zone occurs without any advance of the crack tip, the size of the end zone becomes smaller, tending to a limit value for a given level of load and, when the critical size of the end zone is reached, there is a transition to a fracture process, which corresponds to conditions (4.7) and (4.8) being satisfied.

For the case of monotonic loading of a crack filled with bonds, graphs of the subcritical external load (4.14) and critical external load (corresponding to the state of limit equilibrium) against the relative length of the part of the crack which has no bonds $\lambda_0 = (l - d)/d_0$, where l is the length of the crack together with the end zone, are shown in Fig. 4. For example, curve *1*, determined by Eq. (4.14), corresponds to a change in the external load which is necessary for a decrease in the size of the end zone d (without changing the crack length l), as a result of the rupture of bonds, from a value $d = l$ to a value of $d = d_{cr}$. At the same time, the parameter λ_0 varies in the following range

$$0 \leq \lambda_0 \leq \lambda_m = (l_{cr} - d_{cr})/d_0 = 0.464$$

At the point s_1 (a relative crack length $\lambda_{cr} = l_{cr}/d_0 = (\lambda_m + d_{cr}/d_0) = 0.649$ corresponds to this point), where curve *1* intersects the limit equilibrium curve, which is determined by the solution of Eqs. (4.7) and (4.8), the size of the end zone reaches the critical value $d\text{cr} = d_0(\lambda_{cr} - \lambda_m)$ and a transition to quasistatic growth of the crack occurs.

The critical size of the end zone of the crack in the limit equilibrium state can be obtained both from the system of Eqs. (4.7) and (4.8) as well as from expressions (4.13) and (4.14). The external loads, determined in the state of limit equilibrium from the energy condition (4.13) and the kinematic condition (4.14), must be identical. We obtain the corresponding equation by equating the expressions for the stresses (4.13) and (4.14)

$$B(t)\sqrt{A^2(t) + B^2(t) + 4C(t) + 2\eta R_0} = R_0 + C(t) - B(t)[A(t) - B(t)] \tag{4.15}$$

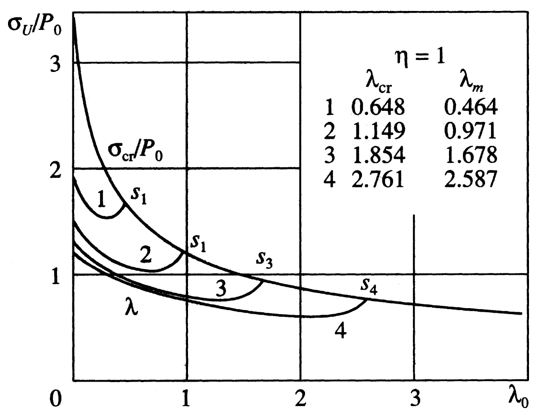

Fig. 4.

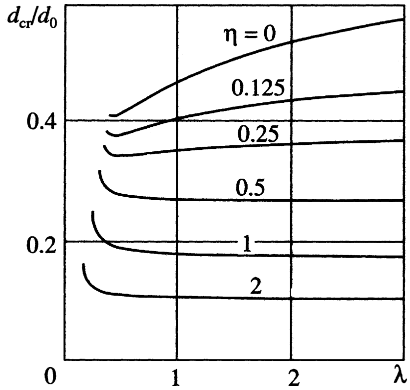

Fig. 5.

Eq. (4.15) can be considered as a non-linear algebraic equation for determining the relative magnitude of the end zone of the crack $t = t_{cr}$ (the parameters R_0 and η are specified). On the other hand, the quantity R_{cr}, corresponding to $t = t_{cr}$,

$$R_{cr} = \Sigma(t) + \sqrt{\Sigma^2(t) + \Psi(t)} \tag{4.16}$$

where

$$\Sigma(t) = B(t)[A(t) - B(t)(1-\eta)] - C(t)$$

$$\Psi(t) = B^2(t)[A^2(t) + B^2(t) + 4C(t)] - [C(t) - B(t)(A(t) - B(t))]^2$$

can be found from Eq. (4.15) for given values of the parameter η of the relative size of the end zone of the crack $t = t_{cr}$.

After calculating the parameter $R_{cr}(t_{cr})$ (see the first formula of (4.9)) from expression (4.16), we determine the crack length, corresponding to the parameter t_{cr}, and the size of the end zone in the state of limit equilibrium

$$l_0 = \frac{\pi E \delta_{cr}}{8 P_0 R_{cr}(t_{cr})} \quad d_{cr} = l_0 t_{cr} \tag{4.17}$$

Graphs of the length of the end zone of the crack in the state of limit equilibrium d_{cr}/d_0 (see formula (4.10)) against the crack length $\lambda = l/d_0$ (one can also write $\lambda = 1/R_{cr}$), obtained from relations (4.17), are shown in Fig. 5 for different values of the parameter η. In the case of a matrix with a relatively high fracture toughness $\eta \geq 1/2$, the length of the end zone of the crack in the state of limit equilibrium decreases as the crack length increases and rapidly tends to a limit value, which is independent of the crack length. In the case of a matrix with a relatively low fracture toughness, the length of the end zone of the crack in the state of limit equilibrium hardly changes in the case of relatively short cracks and then increases, tending to a limit value which is independent of the crack length. If the fracture toughness of the matrix can be neglected ($\eta = 0$), the size of the end zone of the crack in the state of limit equilibrium increases as the crack length increases, tending to the constant value $d_{cr}/d_0 = 1$ when $\lambda \to \infty$. Note that the case $\eta = 0$ when $\lambda \to \infty$ is analogous to the Leonov - Panasyuk model[6] in which the critical size of the end zone decreases as the crack becomes longer, tending to the same limit value.

We next consider the growth of a crack of half length l with an end zone of dimension $0 < d_i < l$ partially filled with bonds. The values of the parameters R_0 and η completely determine the solution of the problem of the limit equilibrium of the crack (see the system of Eqs. (4.7), (4.8)). The manner in which the crack develops under monotonic loading changes, depending on the relation between the specified size of the end zone d and the critical size of this zone in the state of limit equilibrium d_{cr}. In this case, the development of the crack is analysed using relations (4.13) and (4.14) (see Fig. 6).

If the size of the end zone of the crack d_k is specified such that

$$d_{cr} < d_k < l, \quad t_{cr} < t_k < 1 \tag{4.18}$$

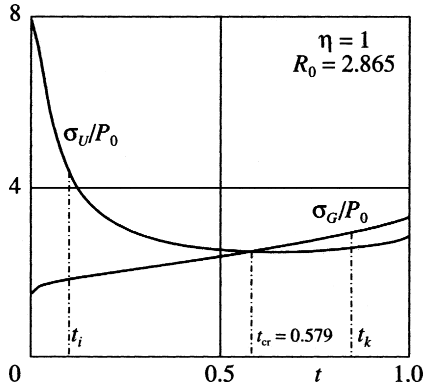

Fig. 6.

then $\sigma_U < \sigma_G$ (see Fig. 6) and, when the external load $\sigma_0 = \sigma_U$ increases, Conditions 1b and 2b are satisfied, that is, the rupture of the bonds at the trailing edge of the end zone occurs and the size of the end zone of the crack decreases without the crack tip advancing. When the end zone reaches its critical size, there is a transition to a quasistatic fracture process described by conditions (3.13).

If, however, the size of the end zone of the crack d_i is specified such that

$$d_i < d_{cr}, \quad t_i < t_{cr} \tag{4.19}$$

then $\sigma_U > \sigma_G$ (see Fig. 6) and, when the external load $\sigma_0 = \sigma_G$ is increased, Conditions 1a and 2a are satisfied, that is, the crack tip advances without the rupture of bonds at the trailing edge of the end zone, and the length of the crack and the length of its end zone increase. When the crack and its end zone reach their critical dimensions, there is a transition to the quasistatic fracture process described by conditions (3.13).

The critical length of a crack, up to which the tip of the initial crack advances, is determined in this case from an equation analogous to (4.15). Suppose that, as a result of quasistatic development up to the state of limit equilibrium, the crack length has increased by Δd and attained a magnitude $l_n = l + \Delta d$ and that the length of the end zone has become $\bar{d}_{cr} = d_i + \Delta d$. Then, the relative size of the end zone in the state of limit equilibrium is

$$\bar{t}_{cr} = \frac{\bar{d}_{cr}}{l_n} = \frac{d_i + \Delta d}{l + \Delta d} = \frac{t_i + \xi}{1 + \xi} \tag{4.20}$$

where $t_i = d_i/l$ is the relative length of the end zone of the crack up to the start of loading (see the second condition of (4.19)), $\xi = \Delta d/l$ is the increment in the length of the end zone, normalized to the initial length of the crack. The parameters R_0 and R_n

$$R_0 = \frac{\pi E \delta_{cr}}{8 P_0 l}, \quad R_n = \frac{\pi E \delta_{cr}}{8 P_0 (l + \Delta d)} = \frac{R_0}{1 + \xi} \tag{4.21}$$

correspond to the initial crack of length l and to a crack of length $l_n = l + \Delta d$.

Substituting (4.20) $t = \bar{t}_{cr}$ into Eq. (4.15) and replacing the parameter R_0 by R_n according to (4.21), we obtain an equation for determining the parameter ξ

$$\begin{aligned} &B(\bar{t}_{cr}(\xi))\sqrt{A^2(\bar{t}_{cr}(\xi)) + B^2(\bar{t}_{cr}(\xi)) + 4C(\bar{t}_{cr}(\xi)) + 2\eta\frac{R_0}{1+\xi}} = \\ &= \frac{R_0}{1+\xi} + C(\bar{t}_{cr}(\xi)) - B(\bar{t}_{cr}(\xi))[A(\bar{t}_{cr}(\xi)) - B(\bar{t}_{cr}(\xi))] \end{aligned} \tag{4.22}$$

After determining the parameter ξ from this equation, we find the increment of the length Δd and the length of the crack l_n after quasistatic growth

$$\Delta d = \xi l, \quad l_n = l(1+\xi)$$

and we determine the relative length of the end zone of the crack from relation (4.20).

Example. There is a crack of length $2l$ and a parameter $R_0 = 2.865$ to which a crack with an end zone of size $t_{cr} = d_{cr}/l = 0.579$ corresponds. Suppose the initial size of the end zone of the crack which is actually specified is $t_i = d_i/l = 0.1 < t_{cr}$ (Fig. 6). Under monotonic loading, the advance of the crack tip starts at $\sigma_G/P_0 = 1.823$, $\sigma_G < \sigma_U$ (see Eq. (4.13) when $R_0 = 2.865$, $t = t_i = 0.1$. The relative change in the crack length is found from the solution of Eq. (4.22): $\xi = \Delta d/l = 0.442$. The remaining parameters are found from relations (4.20) and (4.21)

$$R_n = \frac{R_0}{1+\xi} = 1.99 \quad \bar{t}_{cr} = \frac{\bar{d}_{cr}}{l_n} = \frac{t_i+\xi}{1+\xi} = 0.376$$

The critical external load after the growth of the crack, in the state of limit equilibrium is $\sigma_{cr}/P_0 = 1.952$ (see Eqs. (4.13) or (4.14) when $R_0 = 1.99$, $t = t_{cr} = 0.376$.

We define the "observed" fracture toughness of the material K_c^{ext} and the critical value corresponding to it, that is, the specific fracture energy G_c^{ext} as

$$K_c^{ext} = \sigma_{cr}\sqrt{\pi l}, \quad G_c^{ext} = (K_c^{ext})^2/E` \tag{4.23}$$

These quantities are calculated after solving Eqs. (4.7) and (4.8). In the given case, the parameters (4.23) are not constants of the material and depend on the crack length. The values of G_c^{ext}/G_b (see the last equation of (4.9)) are shown in Fig. 7. As the crack length λ increases, the specific fracture energy tends to a constant value $G_b + G_m$, which is determined by the bond properties in the end zone of the crack and by the matrix material.

The cases of the development of a crack from an initial cut and the zones in which the bonds have been weakened considered above enable us to describe the behaviour of a crack with an end zone under monotonic loading.

4.1. Analysis of limiting cases

We will now consider the behaviour of a crack filled with bonds ($t = (d/l) = 1$ in Eqs. (4.7) and (4.8)) in the case of monotonic loading. When $t = 1$, Eqs. (4.7) and (4.8) take the form

$$\xi^2 - 2\xi\left(1-\frac{2}{\pi}\right) - \frac{4}{\pi}\left(1+\frac{2\eta R_0}{\pi}\right) = 0, \quad \xi = \frac{\sigma_m}{P_0} - 1, \quad \frac{\sigma_{cr}}{P_0} = 1 + \frac{2R_0}{\pi} \tag{4.24}$$

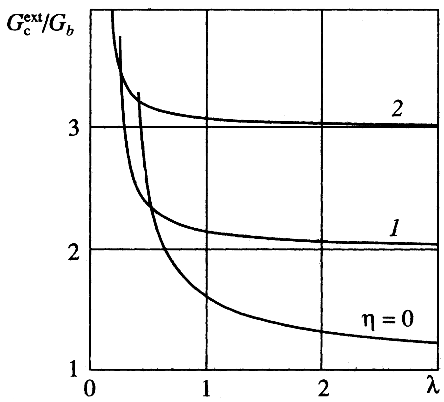

Fig. 7.

where σ_m is the critical external load under which the crack tip begins to advance and σ_{cr} is the critical external load under which rupture of the bonds at the centre of the crack begins.

From the first equation of (4.24) (also, see formula (44.13) when $t=1$), we obtain

$$\frac{\sigma_m}{P_0} = 2\left(1-\frac{1}{\pi}\right)+\sqrt{1+\frac{4}{\pi^2}(1+2\eta R_0)} \tag{4.25}$$

The relation between the values of the stresses σ_{cr} and σ_m is determined by the parameters $\eta=\text{const}$ and R_0, and the magnitude of R_0 decreases as the crack grows. If $\sigma_{cr}>\sigma_m$, then, in the case of a crack filled with bonds, an increase in the external load up to the value $\sigma_0=\sigma_m$ leads to the onset of the advance of the crack tip without the rupture of bonds. If, however, $\sigma_{cr}\le\sigma_m$, then increasing the external load up to the value $\sigma_0=\sigma_{cr}$ leads to the onset of the rupture bonds at the centre of the crack and to the formation of a bond-free zone.

We will now determine the critical value of the parameter $R_0=R_f$ at which $\sigma_{cr}=\sigma_m=\sigma_f$. From equality (4.25) we obtain a quadratic equation in R_f with a positive root

$$R_f = \frac{\pi}{2}\left[\left(1+2\frac{\eta-1}{\pi}\right)+\sqrt{\left(1+2\frac{\eta-1}{\pi}\right)^2+\frac{4}{\pi}}\right] \tag{4.26}$$

(also, see expression (4.16) when $t=1$). When $\eta\gg 1$, it follows from equality (4.26) that $R_f\approx 2\eta$.

For the value $R_0=R_f$, we obtain from relation (4.25) that

$$\frac{\sigma_f}{P_0} = 2\left(1+\frac{\eta-1}{\pi}\right)+\sqrt{\left(1+2\frac{\eta-1}{\pi}\right)^2+\frac{4}{\pi}},\quad \frac{\sigma_f}{P_0}\approx\begin{cases}2.549, & \eta=0\\ 3.508, & \eta=1\end{cases} \tag{4.27}$$

We note that, in the case being considered, the crack is a zone of weakened bonds in the material and that there is no initial cut (crack). Hence, expressions (4.27) give an estimate of the strength of the “defect-free” material, that is, when there is no cut (crack).

In the case of fixed parameters E, P_0, δ_{cr}, expression (4.26) enables us to estimate the length l_f of the initial crack at which the onset of the quasistatic growth of a crack filled with bonds with conditions (4.7) and (4.8) being satisfied, is possible

$$l_f = \frac{\pi E\delta_{cr}}{8P_0R_f} \tag{4.28}$$

On the other hand, the parameters of the bonds (δ_{cr}, P_0) can be chosen in such a way that condition (4.28) is satisfied for a crack of specified size.

When $t\to 0(A(t)\approx\sqrt{2t},\ B(t)\approx\sqrt{2t},\ C(t)\approx -t)$, Eqs. (4.7) and (4.8) take the form

$$(Z_0\xi)^2-2Z_0\xi-2\eta R_0Z_0^2+1=0,\quad Z_0(\xi^2/2+R_0)-\xi=0,\quad \xi=\sqrt{2d_{cr}/l} \tag{4.29}$$

whence it follows that, in the case of cracks with a short end zone, the critical size of the end zone d_{cr} is independent of the crack length:

$$d_{cr} = l\xi^2/2 = d_0(\sqrt{\eta+1}-\sqrt{\eta})^2 \tag{4.30}$$

In the case of a crack with a short end zone, we obtain the critical external load in the limit equilibrium state from relations (4.29)

$$\sigma_{cr} = \sqrt{(1+\eta)\frac{EP_0\delta_{cr}}{\pi l}} = \sqrt{\frac{E(G_b+G_m)}{\pi l}} \tag{4.31}$$

When $\eta\to\infty$ (the fracture toughness of the matrix is much greater than the fracture toughness of the fibres, ($G_m\gg G_b$), it follows from expression (4.30) that

$$d_\infty = d_{cr} = \frac{d_0}{4\eta} = \frac{\pi E\delta_{cr}G_b}{32P_0G_m} = \frac{\pi E\delta_{cr}^2}{32G_m} \tag{4.32}$$

The magnitude of the critical external load σ_∞ when $\eta \to \infty$ is found from expression (4.31)

$$\sigma_\infty = \sigma_{cr} = \sqrt{\frac{EG_m}{\pi l}} \tag{4.33}$$

Expressions (4.30)–(4.33) for the size of the end zone and the magnitude of the critical load are identical to the well-known expressions[9] obtained using a force fracture criterion for a crack with a short end zone.

In the case of a short end zone, we consider, taking account of (4.30) and (4.31), each of the energy characteristics of the crack in the limit equilibrium state separately. The energy consumption rate in the deformation of the bonds (see the first expression of (4.5)) when $t \to 0$ will be ($\varphi(t) \to 0$)

$$G_{bond}(d_{cr}, l) = G_m \tag{4.34}$$

On the other hand, it is possible, starting from expression (3.11) and taking account of relations (4.1), to write

$$G_{bond}(d_{cr}, l) = 2P_0 \int_{l-d_{cr}}^{l} \frac{\partial u(x)}{\partial l} dx - G_b + G_m, \quad G_b = \int_0^{\delta_{cr}} q_y(u) du = P_0 \delta_{cr} \tag{4.35}$$

It follows from equalities (4.34) and (4.35) that

$$2P_0 \int_{l-d_{cr}}^{l} \frac{\partial u(x)}{\partial l} dx = \int_0^{\delta_{cr}} q_y(u) du \tag{4.36}$$

that is, in the case of a short end zone of a crack, the increment in the deformation energy of the bonds in the limit equilibrium state is equal to the density of the deformation energy released during the fracture of the bonds at the trailing edge of the end zone.

Taking account of expressions (4.30) and (4.31), for the deformation energy release rate in the limit equilibrium state of the crack (4.3), we obtain

$$G_{tip}(d_{cr}, l) = \frac{(K_\infty - K_b)^2}{E} = \eta G_b = G_m \tag{4.37}$$

whence the force fracture condition (1.2)[9]

$$K_\infty - K_b = K_{Ic}, \quad K_{Ic} = \sqrt{EG_m} = \sqrt{\eta E P_0 \delta_{cr}} \tag{4.38}$$

follows.

Hence, in the case of a short end zone, the first energy condition of (3.13) splits into two independent Eqs. (4.34) and (4.37), the first of which determines the contribution made to the fracture toughness of the material by the bonds and the second of which reduces to a force condition for the advance of the crack tip.

In the case of a matrix with a low fracture toughness $\eta = 0$, the condition for the stresses at the crack tip to be finite

$$K_\infty - K_b = 0 \tag{4.39}$$

follows from equalities (4.37). Hence, in the case of a short end zone of a crack and neglecting the fracture toughness of the matrix, we obtain a condition which is the initial assumption in the Panasyuk and Barenblatt models.

4.2. Comparison of the energy and force fracture criteria

We will now carry out a comparative analysis of the fracture criterion considered above for a crack with bonds in the end zone (subsequently referred to as the energy criterion) and the fracture criterion with a force condition for the advance of the crack tip[9] (subsequently referred to as the force criterion) for a problem with constant stresses in the end zone.

We shall assume that the conditions for the rupture of the bonds at the trailing edge of the end zone are identical in both criteria and are determined by the second equation of (3.13). The equation for the force fracture criterion when there are constant stresses in the end zone, which is analogous to the first of conditions (3.13), has the form (4.38). We will now assume that this condition holds for any size of the end zone.

Using expressions (4.3) and (4.6), we convert condition (4.38) to the dimensionless form

$$1/Z_0 - A(t) = \sqrt{2\eta R_0}, \quad t = d/l \tag{4.40}$$

We obtain the equation for determining the length of the end zone of the crack in the limit equilibrium state in accordance with the force fracture criterion from Eqs. (4.40) and (4.8), on eliminating the parameter Z_0 from them. We have

$$C(t) - B(t)\sqrt{2\eta R_0} + R_0 = 0, \quad t = d_{cr}/l \tag{4.41}$$

After solving this equation, the critical external load can be determined, for example, from an expression analogous to (4.14).

On the other hand, by analogy with Eqs. (4.15), (4.41) can be considered as an equation for determining the parameter $R_{cr} = R_0$ for a specified value of $t_{cr} = d_{cr}/l$

$$\xi^2 - \xi B(t_{cr})\sqrt{2\eta} + C(t_{cr}) = 0, \quad \xi = \sqrt{R_{cr}} \tag{4.42}$$

From Eq. (4.42), we obtain

$$R_{cr} = [B(t_{cr})\sqrt{\eta} + \sqrt{\eta B^2(t_{cr}) - 2C(t_{cr})}]^2/2 \tag{4.43}$$

When $t_{cr} \to 0$, from the solution (4.43), we have

$$R_{cr} = t_{cr}(\sqrt{\eta + 1} - \sqrt{\eta})^{-2} \tag{4.44}$$

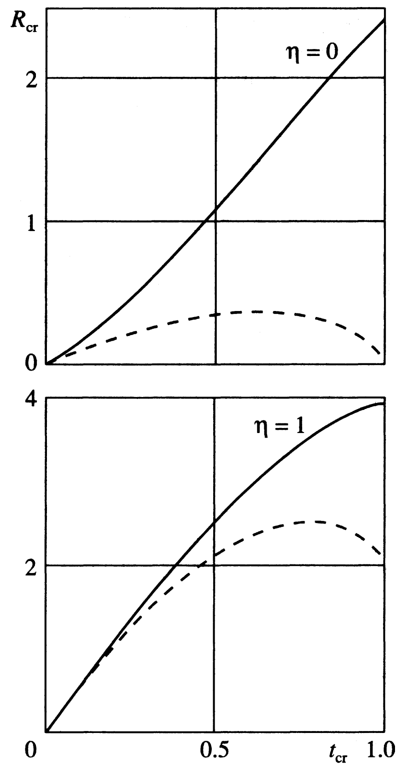

Fig. 8.

An expression for R_{cr}, which is identical to (4.44), follows from expression (4.16) when $t_{cr} \to 0$, which is evidence of the equivalence of the fracture criteria being considered in this case.

For a comparative analysis of the energy and force fracture criteria, we will consider relations (4.15) and (4.43) for the parameters $R_{cr} = d_0/l_{cr}$ for these fracture criteria as a function of the relative length of the end zone of a crack t_{cr} in the limit equilibrium state. For small values of t_{cr}, both criteria give close results (see Fig. 8, $\eta = 1$), and the difference increases as the relative size of the end zone increases. Note that, in the case of a fixed relative size of the end zone, the energy criterion gives a greater value of the parameter R_{cr} than the force criterion, which, in its turn, corresponds to a shorter crack and a greater critical external stress. The increase in the critical load when the energy criterion is used is explained by taking account of the work done in deforming the bonds. When $\eta \to \infty$, both criteria give similar results for $0 < t_{cr} \leq 1$ and $R_{cr} \to 2\eta$ when $t \to 1$.

The results in the case of a matrix with a low fracture toughness ($\eta = 0$) are fundamentally different (see Fig. 8). In the case of a short end zone the results are close but, already when $t_{cr} > 0.1$, a considerable divergence is observed. Note that the force criterion is inapplicable in the case of a crack filled with bonds when $\eta = 0$, since expression (4.1), which has been written taking account of the finiteness of the stresses (4.39), gives a zero opening of the crack.

In Fig. 8, when $t_{cr} = 1$, we have $R_{cr} = 0$ in the case of the force fracture criterion, which formally corresponds to a crack of infinite length and, correspondingly, to an end zone of infinite size. The maximum value of the parameter $R_{cr}^m \approx 0.368$ is reached when $t_{cr} \approx 0.632$. When $R_{cr} > R_{cr}^m$, cracks, which satisfy the limit equilibrium conditions, do not exist within the framework of the force fracture criterion.

Hence, the energy and force criteria for the development of a crack give close estimates of the fracture parameters in the case of crack with a short end zone and, also, in the case of a composite material with a matrix possessing a high fracture toughness.

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